Highly accurate κ-μ approximation to sum of M independent non-identical Ricean variates

J.C.S. Santos Filho and M.D. Yacoub

A scheme to approximate the distribution of the sum of M independent, non-identically distributed Ricean random variables by the κ-μ distribution is proposed. To this end, appropriate κ-μ distribution parameters are derived. The summands are assumed to have arbitrary mean powers and arbitrary fading parameters. The differences between exact and approximate distribution curves are almost imperceptible.

Introduction: Several distributions have been proposed to describe the fading statistics of the wireless channel. In particular, the Rice distribution has proven very suitable in modelling the environment in which a line-of-sight propagation condition exists. It includes the Rayleigh distribution as a special case and can be approximated by the Nakagami-m distribution [1].

The performance analysis of many important practical wireless applications involve the sum of M fading random variables (RVs), including equal-gain combining, signal detection, linear equalisers, outage probability, intersymbol interference, phase jitter, and error bounds calculations for coding in satellite communications [2]. Unfortunately, the exact statistical treatment of this problem is rather intricate. For independent fading RVs, the resultant sum probability density function (pdf) emerges as the convolution of the individual pdf's of the summands. Based on [3], a finite-range multifold integral solution was proposed in [4] considering independent identically distributed (i.i.d.) Nakagami-m summands. In [5 and 6], an approximate infinite series technique was presented for computing the pdf of the sum of independent Ricean and Nakagami-m RVs, respectively.

The inherent intricacies of the above methods have been circumvented by simpler approximate solutions for the Nakagami-m case. In [1], the pdf of the sum of M i.i.d. Nakagami-m RVs was approximated by a Nakagami-m pdf. Anchored in that idea, the parameters of the approximate Nakagami-m distribution to the sum of two correlated, identically distributed Nakagami-m RVs were obtained in [7]. More recently, a Nakagami-m approximation to the sum of M independent non-identically distributed (i.n.d.) Nakagami-m RVs was derived [8].

In [9] a general fading model, the κ-μ distribution, which includes the Rice and the Nakagami-m distributions as special cases, has been presented. In this Letter, we propose to approximate the pdf of the sum of M i.n.d. Ricean RVs by a κ-μ pdf. To this end, appropriate κ-μ distribution parameters are derived. The summands are assumed to have arbitrary mean powers and arbitrary fading parameters. As is illustrated, the differences between exact and approximate distribution curves are almost imperceptible.

κ-μ distribution revisited: The pdf of a κ-μ RV $\tilde{R}$ is given by [9]

$$p_{\tilde{R}}(r) = \frac{2\mu(1+\kappa)^{(\mu+1/2)}r^{\mu}}{\kappa^{(\mu-1/2)}\Omega^{(\mu+1/2)}}\exp\left(\mu\left[-\kappa-\frac{(1+\kappa)r^2}{\Omega}\right]\right) \times I_{\mu-1}(2\mu\sqrt{\kappa(1+\kappa)/\Omega}\,r) \quad (1)$$

where

$$\Omega = E[\tilde{R}^2] \quad (2)$$

is the mean power of $\tilde{R}$, $\kappa \geq 0$ is the ratio between the total power of the dominant components and the total power of the scattered waves,

$$\mu = \frac{\Omega^2}{(E[\tilde{R}^4]-\Omega^2)}\frac{(1+2\kappa)}{(1+\kappa)^2} \quad (3)$$

and $I_\nu(\cdot)$ is the modified Bessel function of the first kind and νth order ($E[\cdot]$ denotes expectation). Although not presented in [9], a moment-based estimator for κ can also be derived, as follows. From (1), the nth moment of $\tilde{R}$ is calculated as

$$E[\tilde{R}^n] = \frac{\Gamma(\mu+n/2)\exp(-\kappa\mu)\Omega^{n/2}}{\Gamma(\mu)[(1+\kappa)\mu]^{n/2}}\,{}_1F_1\left(\mu+\frac{n}{2};\mu;\kappa\mu\right) \quad (4)$$

where $\Gamma(\cdot)$ is the gamma function and ${}_1F_1(\cdot;\cdot;\cdot)$ is the confluent hypergeometric function. Replacing (3) into (4) for $n=6$, and after some algebraic manipulations, κ can be finally written as

$$\kappa^{-1} = \frac{\sqrt{2}(E[\tilde{R}^4]-\Omega^2)}{\sqrt{2E^2[\tilde{R}^4]-\Omega^2 E[\tilde{R}^4]-\Omega E[\tilde{R}^6]}} - 2 \quad (5)$$

Note that, knowing $E[\tilde{R}^2]$, $E[\tilde{R}^4]$, and $E[\tilde{R}^6]$, the distribution parameters Ω, κ, and μ of $\tilde{R}$ are uniquely established by (2), (5) and (3), respectively.

In a wider context, the moment-based estimators (2), (3) and (5) can be used to calculate the parameters Ω, κ and μ of any arbitrary (not necessarily κ-μ distributed) RV, say X, simply replacing $\tilde{R}$ by X in the referred formulae. In such a case, $E[X^2]$, $E[X^4]$ and $E[X^6]$ are required to accomplish the calculations. For instance, if X is Nakagami-m distributed with mean power Ω_N and Nakagami-m fading parameter m, then, using (2), (3) and (5) as described, we obtain $\Omega=\Omega_N$, $\kappa=0$ and $\mu=m$. In the same way, if X is Rice distributed with mean power Ω_R and Rice fading parameter k, then $\Omega=\Omega_R$, $\kappa=k$ and $\mu=1$. And so on for any other RV. This approach is explored in the next Section to compute the parameters Ω, κ and μ for the sum of independent Ricean RVs.

Proposed Approximation: Let R be the sum of M i.n.d. Ricean RVs R_i, $i=1,\ldots,M$,

$$R = \sum_{i=1}^{M} R_i \quad (6)$$

so that the pdf of R_i is given by

$$p_{R_i}(r_i) = \frac{2(1+k_i)r_i}{\Omega_i}\exp\left(-k_i-\frac{(1+k_i)r_i^2}{\Omega_i}\right)I_0(2\sqrt{k_i(1+k_i)/\Omega_i}\,r_i) \quad (7)$$

where $\Omega_i = E[R_i^2]$ is the mean power of R_i, k_i is the Ricean fading parameter, and $I_0(\cdot)$ is the modified Bessel function of the first kind and zeroth order. Next, we propose an approximation to $p_R(\cdot)$, the exact pdf of R.

The physical model for the κ-μ distribution considers a signal composed of clusters of multipath waves propagating in an homogeneous environment, with each cluster having its own dominant and scattered wave components. In fact, it can be recognised from [9, equation (3)] that the squared κ-μ RV is formulated as the sum of independent squared Ricean RVs. Inspired by this, we propose to approximate $p_R(\cdot)$ by $p_{\tilde{R}}(\cdot)$, with the latter given in (1). An analogous motivation seems to have supported a Nakagami-m approximation to the sum of Nakagami-m RVs in [1, 7, 8]. The question now is to find Ω, κ, and μ that render (1) a good approximation to $p_R(\cdot)$. Our suggestion is to use the parameters Ω, κ and μ of R itself. As mentioned at the end of the preceding Section, these parameters are calculated replacing $\tilde{R}$ by R in (2), (3), and (5). The required moments $E[R^2]$, $E[R^4]$ and $E[R^6]$ can be evaluated from [8]

$$E[R^n] = \sum_{n_1=0}^{n}\sum_{n_2=0}^{n}\cdots\sum_{n_{M-1=0}}^{n_{M-2}}\binom{n}{n_1}\binom{n_1}{n_2}\cdots\binom{n_{M-2}}{n_{M-1}} \times E[R_1^{n-n_1}]E[R_2^{n_1-n_2}]\cdots E[R_M^{n_{M-1}}] \quad (8)$$

where the nth moment of each Ricean summand R_i is known to be given by

$$E[R_i^n] = \frac{\Gamma(1+n/2)\exp(-k_i)\Omega_i^{n/2}}{(1+k_i)^{n/2}}\,{}_1F_1\left(1+\frac{n}{2};1;k_i\right) \quad (9)$$

A last remark is required. As already stated, the parameter κ of a κ-μ RV is a non-negative real number. However, for RVs that are not κ-μ distributed, in some cases the estimation of the corresponding κ parameter as defined in (5) may yield improper values. In particular, for sums of Ricean RVs, it has been observed, through exhaustive numerical examples, that complex values with negative real part for the estimated κ may result. In such cases, the use of $\kappa \to 0$ has proven to give excellent results (note that zero is the non-negative real number with the minimum distance from the above-mentioned complex estimation). In effect, as $\kappa \to 0$, the κ-μ distribution reduces to the exact Nakagami-m distribution with fading parameter $m=\mu$ [9].

Comparisons: In this Section, we illustrate with some sample examples how the proposed approximation yields remarkable results. Let $P = R/\hat{r}$ be the normalised Ricean sum, with $\hat{r} = \prod_{i=1}^{M} \Omega_i^{1/(2M)}$. Fig. 1 shows the exact and approximate pdf's $p_P(\cdot)$ of P for i.i.d. Ricean summands with $k_i = 0, 1, 2, 3$ and $M = 2, 3, 4$. The exact approach derived in [4] for i.i.d. Nakagami-*m* summands can be easily extended to calculate the sum pdf for any set of i.n.d. RVs and has been used here in order to support the comparisons. Note that the differences between exact and approximate curves are indeed minimal, vanishing as M and as k_i increase. Non-identical summands are considered in Fig. 2 for $M = 3$. An exponential power decay profile with decay factor δ is used to model the power imbalance, so that $\Omega_i = \Omega_1 \exp(-\delta(i-1))$, $i = 1, 2, 3$. Here again, the exact and approximate curves are almost indistinguishable from each other.

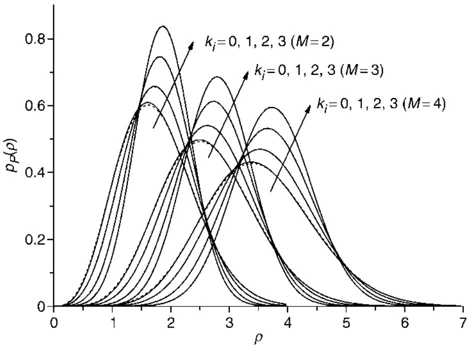

Fig. 1 *pdf of sum of i.i.d. Ricean RVs*
Exact solution: solid; approximate solution: dashed

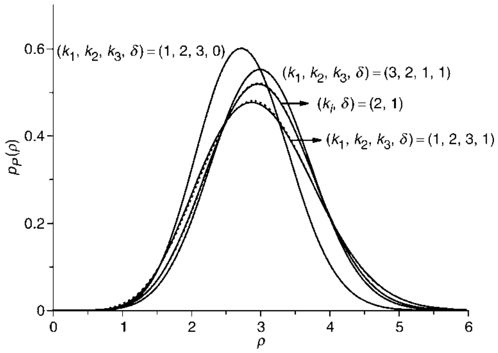

Fig. 2 *pdf of sum of three i.n.d. Ricean RVs*
Exact solution: solid; approximate solution: dashed

Conclusions: A highly accurate κ-μ approximation to the sum of M i.n.d. Ricean RVs having arbitrary mean powers and arbitrary fading parameters is proposed. Our formulation finds applicability in important communications issues such as equal-gain combining, signal detection, linear equalisers, outage probability, intersymbol interference, phase jitter, and error bounds calculations for coding in mobile satellite communications.

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J.C.S. Santos Filho and M.D. Yacoub (*Department of Communications, School of Electrical and Computer Engineering, University of Campinas, PO Box 6101, 13083-852 Campinas, SP, Brazil*)

E-mail: candido@decom.fee.unicamp.br

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